THE THEORY OF PROBABILITY

The theory of probability is a part of Mathematics that deals with studying of the rules of the so-called random events. Just like any other science, it is related with the people's necessities. For example: the meteorological phenomena (the occurrence and amount of rainfall, droughts …), the quality control of serial industrial production (occurrence of wholesome products …), etc. This means that the theory of probability deals with the determination of the probability of the occurrence of random events (phenomena).

Random events

It is known that the study of nature and society or the world in general, occurs as a result of various tests – experiments or observations.

An experiment presents the realization of a certain set of conditions.

An event is the result or the outcome of an experiment. (A, B, C …)

Example 1: We have performed an experiment by throwing a coin in the air. While the coin is falling on the table, this can happen: event A: "heads" appeared on the upper side of the coin or event B: "tails" appeared on the upper side.

Example 2: While throwing a dice on the table, he following events can happen: event A: a point appeared on the upper side of the dice; event B: two points appeared on the upper side of the dice; event C: three points appeared on the upper side; event D: four points appeared on the upper side; event E: five points appeared on the upper side; event F: six points appeared on the upper side of the dice.

Example 3: When throwing a blue and a red colored dice, every possible "event" could be represented as an ordered pair (x, y), where x is the number of points shown on the upper side of the blue dice, while y is the number of points that appeared on the upper side of the red dice. All possible events that would appear during this experiment can be described by the set $\Omega = \{(x,y)\ \ x,y = 1,2,3,4,5,6\}$.

Elementary events are only the possible and the equally possible events of an experiment (E_1, E_2, E_3 ..... E_n).

The space of elementary events is the set of all elementary events $\Omega = \{E1\ , E2\ , E3\ , \ldots\ldots En\}$.

A reliable event is the event that would surely happen under certain circumstances.

An impossible event is the event that would not happen under certain circumstances.

An opposite event of the event A is the event $\bar{A}$ and it happens only when the event A does not occur.

Disjunctive events are events that cannot occur at the same time.

Example 4: Experiment: Throwing of a dice

Events: A: A number from 1 to 6 appeared

B: Number 7 appeared

C: An even number occurred

D: An odd number occurred

Discussion: The event A is a reliable event, while event B is an impossible event, and the events C and D are disjunctive events.

Task 1: You can choose a number from a set of single-digit numbers. The following events can occur:

A: You can select a number that is divisible by 2.

B: You can select a number that is divisible by 3.

C: You can select an odd number.

D: You can select an odd number that is divisible by 3.

E: You can select an odd number that is divisible by 2.

Describe the space of elementary events as well as the events A, B, C, D, and E.

Solution: The elementary events are formed:

E_1: Number 1 is selected E_2: Number 2 is selected E_3: Number 3 is selected

E_4: Number 4 is selected E_5: Number 5 is selected E_6: Number 6 is selected

E_7: Number 7 is selected E_8: Number 8 is selected E_9: Number 9 is selected

So the space of elementary events is $\Omega = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9\}$

The events are: $A = \{E_2, E_4, E_6, E_8\}$ $B = \{E_3, E_6, E_9\}$ $C = \{E_1, E_3, E_5, E_7, E_9\}$

$D = \{E_3, E_9\}$ $E = \emptyset$

Task 2: The following events occur when you throw simultaneously the two different colored dice:

A: The sum of the points that appeared on both upper sides is an even number.

B: The sum of the points that appeared on both upper sides is bigger than 12.

C: An equal number of points appear on both dice.

D: Odd numbers of points appear on both dice.

Describe the space of elementary events as well as the events A, B, C, and D.

Solution: $\Omega = \{(x, y)\ \ x, y = 1,2,3,4,5,6\}$

$$A = \begin{Bmatrix} (1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), \\ (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \end{Bmatrix}$$

$$B = \emptyset \qquad C = \begin{Bmatrix} (1,1), (2,2), (3,3), \\ (4,4), (5,5), (6,6) \end{Bmatrix}$$

$$D = \begin{Bmatrix} (1,1), (1,3), (1,5), (3,1), (3,3), (3,5), \\ (5,1), (5,3), (5,5) \end{Bmatrix}$$

Classical definition of probability

In life, we often say that today it will probably rain or some products are probably of good quality.

Thus, we perform a subjective evaluation of the ability to occur an event, based on the data we have.

The magnitude that objectively characterizes the ability to occur an event, under certain circumstances that can be repeated many times is called **probability** (mathematical probability).

Example 1: There are 10 equal sized balls in a box but 3 are white and 7 are red. The random draw of a ball is called an elementary event. If we want to draw a red ball and if we really do that, then the event is called **favorable event** "A". If we want to draw a red ball but instead a white ball appears, then the event is called **unfavorable event (diverse event)** "$\overline{A}$". There are 7 opportunities for a favorable event A to be performed and 3 opportunities for an unfavorable event to be performed $\overline{A}$ out of 10 possible events.

Definition: If 'n' is the number of all elementary (equally possible) events and 'm' is the number of all possible events of the favorable event A, then the quotient of these two numbers represents the **probability of the event A**

$$\boldsymbol{P}(\mathrm{A}) = \frac{\boldsymbol{m}}{\boldsymbol{n}} \qquad (0 < m < n)$$

This means: $P(\mathrm{A}) = \frac{7}{10}$ $\quad P(\overline{\mathrm{A}}) = \frac{3}{10}$

Characteristics of probability

1. $\mathbf{0} < \boldsymbol{P}(\mathrm{A}) < \mathbf{1}$
2. If the event is reliable, then $P(\mathrm{A}) = 1$
3. If the event is impossible, then $P(\mathrm{A}) = 0$
4. If the event $\overline{\mathrm{A}}$ is opposite to the event A, then $P(\overline{\mathrm{A}}) = 1 - P(\mathrm{A})$

Task 3: A person has forgotten the last two digits of his friend's phone number. That person only remembers that they are different digits. What is the probability to guess the digits that are necessary?

Solution: The number of all possible elementary events is equal to the number of all variations without repetitions of the class 2 of 10 elements (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

This means: $V_{10}^2 = \frac{10!}{(10-2)!} = \frac{10 \cdot 9 \cdot 8!}{8!} = 10 \cdot 9 = 90$ (n=90)

Only one favorable event is reliable out of these 90 elementary events and that is the event A: the necessary numbers are selected

This means: $P(\text{A}) = \frac{1}{90}$

Task 4: There are 5 white and 4 black balls in a box. Two balls are drawn simultaneously from the box. Determine the probability that the two drawn balls are white.

Solution: We mark the event A: two white balls are drawn. The number of the possible (elementary) events is the combination without repetition out of 9 (5+4) elements class 2 (two drawn balls).

So: $C_9^2 = \frac{9!}{2! \cdot (9-2)!} = \frac{9 \cdot 8 \cdot 7!}{2 \cdot 1 \cdot 7!} = 9 \cdot 4 = 36$ (n=36)

The favorable number of opportunities for the event A to be performed is the combination without repetition of 5 (white) elements class 2.

This means: $C_5^2 = \frac{5!}{2! \cdot (5-2)!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!} = 5 \cdot 2 = 10$ (m=10)

So: $P(\text{A}) = \frac{10}{36}$

The probability of the set of random events:

Let A and B are **disjunctive random events** with corresponding probabilities P(A) and P(B). Then, the probability to be performed for one of the events A or B is: **P(A+B) = P(A) +P(B)**

Example 1: There are 8 red, 6 blue and 14 white balls in a box. What is the probability for a colored ball (a red or a blue one) to be drawn out of the box?

Solution: Event A: A red ball is drawn.

Event B: A blue ball is drawn.

The events A and B are disjunctive events, so the event A+B is the presence either of A or B (at least one of them).

Because $P(\mathrm{A}) = \frac{8}{28} = \frac{2}{7}$ $P(\mathrm{B}) = \frac{6}{28} = \frac{3}{14}$ the result is $P(\mathrm{A} + \mathrm{B}) = \frac{2}{7} + \frac{3}{14} = \frac{1}{2}$

Example 2: A circular target is divided into three concentric zones. The probability to hit the first zone by one bullet is 0.16; the second 0.24; and the third is 0.17. What is the probability to miss the target?

Solution: Let D denotes the miss of the target; A: the hit in the first zone; B: the hit in the second zone; and C: the hit in the third zone.

A person should use only one bullet, the events A, B and C are disjunctive, so the event

$\overline{D}$=A+B+C is the hit in one of those zones. According to this:

P($\overline{D}$) = P(A+B+C) = P(A) + P(B) + P(C) = 0,16 + 0,24 + 0,17 = 0,57

The probability that we were looking for is P(D) = 1 – 0,57 = 0,43.

Let’s suppose that the events A and B are not disjunctive events (they do not exclude mutually). The event A+B is either the presence of the event A or the presence of the event B, or the presence of the event A and the event B (AB).

P(A+B) = P(A) +P(B) – P(AB)

Example 3: What is the probability at throwing of the dice, on the its upper side to occur points whose number is divisible with either 2 or 3?

Solution: There are 6 elementary events (number 1 appeared, number 2 appeared, …

… number 6 appeared), so n=6

Event A: There appeared a number that is divisible with 2 A=$\{2,4,6\}$, so m=3

Event B: There appeared a number that is divisible with 3 B=$\{3,6\}$ so m=2

Because $A \cap B = \{6\}$, it means that A and B are not disjunctive events (these events do not exclude between themselves).

Then, the probability for the event A+B to be realized (a number that is divisible with 2 or 3 appeared) will be:

P(A+B) = P(A) + P(B) – P(AB)

P(A) = $\frac{3}{6} = \frac{1}{2}$ P(B) = $\frac{2}{6} = \frac{1}{3}$ P(AB) = P(A) $\cdot$ P(B) = $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

This means: P(A+B) = $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$

The probability of the product of random events

We say that two random events are mutually independent events only if the occurrence of one event does not affect the odds of the other event.

If the occurrence of one event affects the odds of the other event, then those two events are mutually dependent events.

The probability of the mutual (simultaneous) occurrence of the random events A and B is equal to the product of the probabilities of the corresponding events.

If A and B are independent events, then: P(AB) = P(A) $\cdot$ P(B)

If A and B are dependent events, then: $P(AB) = P(A) \cdot P_A(B)$ or

$$P(AB) = P_B(A) \cdot P(B)$$

Example 1: (independent events)

What is the probability at a simultaneous throwing of a coin and a dice, for the heads and six points to appear?

Solution: Event A: Heads appeared

(2 elementary events n=2, one favorable event m=1, $P(A) = \frac{1}{2}$)

Event B: Six points appeared

(6 elementary events n=6, one favorable event m=1, $P(B) = \frac{1}{6}$)

So: $P(AB) = P(A) \cdot P(B)$ or $P(AB) = \frac{1}{2} \cdot \frac{1}{6}$ or $P(AB) = \frac{1}{12}$

Example 2: (dependent events)

Each letter from the word "MATEMATIKA" is written on a separate piece of paper. The pieces of paper are put in a box and are mixed well. One piece of paper is drawn at a time and after every forth one, all four are returned back in the box. What is probability of the four drawn pieces of paper to form the word "MAMA" set by the order of drawing?

Solution: Elementary events: n=10

Event A_1: A piece of paper with the letter M is drawn (favorable events 2, m=2, n=10)

Event A_2: A piece of paper with the letter A is drawn (favorable events 3, m=3, n=9)

Event A_3: A piece of paper with the letter M is drawn (favorable events 1, m=1, n=8)

Event A_4: A piece of paper with the letter A is drawn (favorable events 2, m=2, n=7)

Thus: $P(A_1) = \frac{2}{10} = \frac{1}{5}$ $P_{A1}(A_2) = \frac{3}{9} = \frac{1}{3}$ $P_{A1,A2}(A_3) = \frac{1}{8}$ $P_{A1,A2,A3}(A_4) = \frac{2}{7}$

Follows: $P(A_1A_2A_3A_4) = P(A_1) \cdot P_{A1}(A_2) \cdot P_{A1,A2}(A_3) \cdot P_{A1,A2,A3}(A_4)$

$$P(A_1A_2A_3A_4) = \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{1}{8} \cdot \frac{2}{7} = \frac{2}{840} = \frac{1}{420}$$

Random variables

If the event: "X receives the value x", i.e. "X=x" is a random event, then X is called a random variable.

If the random variable receives final number of different values, then it is called a **discrete random value**.

If the random variable receives countless different values, then it is called a **continuous random variable**.

Example 1: Let's denote with X the number that represents the sum of the dots which appear at a throwing of two dice. Then, X will be a discrete variable which receives the values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 (for example: we get the value 2 when there is one dot on one dice and there is also one dot on the other dice or (1, 1),, We get the value 3 of (1, 2) and (2, 1),, We get the value 4 of (1, 3), (3, 1) or (2, 2),, ….).

The probabilities of X are as follow: $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$.

The sum of all probabilities that X receives is 1.

The law for allocation of the probabilities of random variables

Let X is a random variable,

$x_1, x_2, \ldots. x_n$ are the values that the variable X receives,

$p_1, p_2, \ldots p_n$ are the appropriate probabilities, where $p_1 + p_2 + \ldots + p_n = 1$.

The set of pairs (x_i, p_i), $i=1,2,\dots n$ represents allocation of the probabilities of the **random variable X.**

This allocation schematically is as follows:

X	x_1	x_2	....	x_n
P(x)	p_1	p_2	....	p_n

The rule, according to which a corresponding value is added to the value of the random probability, is called the **law for allocation of the probabilities of the random variables.**

Example 1: The law for allocation of the probabilities of the random variable X is as follows:

X	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Numerous characteristics of the random variables

For a better and complete determination of a random variable, besides the law for allocation of the probabilities, we need a number of characteristics:

Average and Mathematical expectation of the random variable

In a series of n experiments, the discrete random variable X receives the following values: $x_1, x_2, \dots x_k$ by order $n_1, n_2, \dots n_k$, then the number:

$\overline{X} = \frac{n_1x_1 + n_2x_2 + \dots + n_kx_k}{n}$ where $n_1 + n_2 + \dots + n_k = n$, it is called **average of the random variable.**

If $p_1, p_2, \dots p_k$ are the corresponding probabilities to the values $x_1, x_2, \dots x_k$, then the number: $M(X) = p_1x_1 + p_2x_2 + \dots + p_kx_k$ is called **mathematical expectation (hope) of the random variable.**

When n is a big number, i.e. when the experiment is done with great number of series, the average $\overline{X}$ will be closely equal to the mathematical expectation M(X) of the random variable.

So: $\overline{X} = p_1x_1 + p_2x_2 + ... + p_\kappa x_\kappa$

Example 1: A target has been divided into three zones. When you hit the first zone, you will receive three points, the second will get you 2 points and the third a point. The hitting of a certain zone is a random variable. If two shooters know the laws for allocation of the probabilities of the hitting X and Y, determine which shooter is better?

X	1	2	3
P(x)	0,3	0,2	0,4

y	1	2	3
P(y)	0,1	0,7	0,2

Solution: $\overline{X} = 1 \cdot 0{,}3 + 2 \cdot 0{,}2 + 3 \cdot 0{,}4 = 1{,}9$ $\overline{y} = 1 \cdot 0{,}1 + 2 \cdot 0{,}7 + 3 \cdot 0{,}2 = 2{,}1$

Thus: the second shooter is better than the first one, because he gets at average level greater points in the great number of hitting series.

Average deviation and dispersion of the random value

For example, in ballistics, it is necessary to know how close the ball, shot out of a weapon, falls down to the target under given circumstances. In other words, it is essential to know how the balls disperse in the vicinity of the target. If the target is marked with O, and the places where the balls fall with M_1, M_2, M_3, then the distances $\overline{OM_1}$, $\overline{OM_2}$, $\overline{OM_3}$ represent the deviation of the balls to the target. To assess the efficiency of firing with certain weapon under given circumstances, first we should evaluate the deviation as a random variable.

Let X is a random variable, $\overline{X}$ is its average, x_1, x_2, ...x_κ are the possible values to X, while p_1, p_2 , ... p_κ are the corresponding probabilities.

The random variable X - $\overline{X}$, whose possible values are the differences $\mathbf{x_1} - \overline{X}$, $\mathbf{x_2} - \overline{X}$, ..., $\mathbf{x_к} - \overline{X}$ is called **deviation of the random variable** X form its average $\overline{X}$.

The number: $\lambda = p_1|x_1 - \overline{X}| + p_2|x_2 - \overline{X}| + \dots + p_к|x_к - \overline{X}|$ is called **average deviation of the random variable** X, where p_1, p_2, ... $p_к$ are the probabilities for the values $x_1 - \overline{X}$, $x_2 - \overline{X}$, ..., $x_к - \overline{X}$.

The number: $\delta^2 = (x_1 - \overline{X})^2 \cdot p_1 + (x_2 - \overline{X})^2 \cdot p_2 + \dots + (x_к - \overline{X})^2 \cdot p_к$ is called **dispersion** (scattering) **of the random variable** X.

The number: $\delta = \sqrt{(x_1 - \overline{X})^2 \cdot p_1 + (x_2 - \overline{X})^2 \cdot p_2 + \cdots + (x_к - \overline{X})^2 \cdot p_к}$ is called **average squared deviation of the random variable** X.

Example 2: Determine λ and δ^2 from the previous example with the two shooters

Solution: Average deviation of the variable X is:

$$\lambda_x = |1 - 1{,}9| \cdot 0{,}3 + |2 - 1{,}9| \cdot 0{,}2 + |3 - 1{,}9| \cdot 0{,}4 = 0{,}73$$

Average deviation of the variable Y is:

$$\lambda_y = |1 - 2{,}1| \cdot 0{,}1 + |2 - 2{,}1| \cdot 0{,}7 + |3 - 2{,}1| \cdot 0{,}2 = 0{,}40$$

Dispersion of the variable X is:

$$\delta_x^2 = (1 - 1{,}9)^2 \cdot 0{,}3 + (2 - 1{,}9)^2 \cdot 0{,}2 + (3 - 1{,}9)^2 \cdot 0{,}4 = 0{,}73$$

Dispersion of the variable X is:

$$\delta_y^2 = (1 - 2{,}1)^2 \cdot 0{,}1 + (2 - 2{,}1)^2 \cdot 0{,}7 + (3 - 2{,}1)^2 \cdot 0{,}2 = 0{,}47$$

Average square deviation of the variable X is:

$\delta_x = \sqrt{0{,}73} = 0{,}85$

Average square deviation of the variable Y is:

$\delta_y = \sqrt{0{,}47} = 0{,}68$

This would mean that not only $\lambda_x > \lambda_y$, but $\delta_x > \delta_y$, and even the dispersion of the hits of the first shooter is greater from the dispersion of the hits of the second one. Thus, the second shooter is better than the first one.